C5. A hunter and an invisible rabbit play a game in the Euclidean plane. The hunter's starting point H_0 coincides with the rabbit's starting point R_0. In the n^{th} round of the game ($n \geqslant 1$), the following happens.

(1) First the invisible rabbit moves secretly and unobserved from its current point R_{n-1} to some new point R_n with $R_{n-1}R_n = 1$.

(2) The hunter has a tracking device (e.g. dog) that returns an approximate position R'_n of the rabbit, so that $R_nR'_n \leqslant 1$.

(3) The hunter then visibly moves from point H_{n-1} to a new point H_n with $H_{n-1}H_n = 1$.

Is there a strategy for the hunter that guarantees that after 10^9 such rounds the distance between the hunter and the rabbit is below 100?

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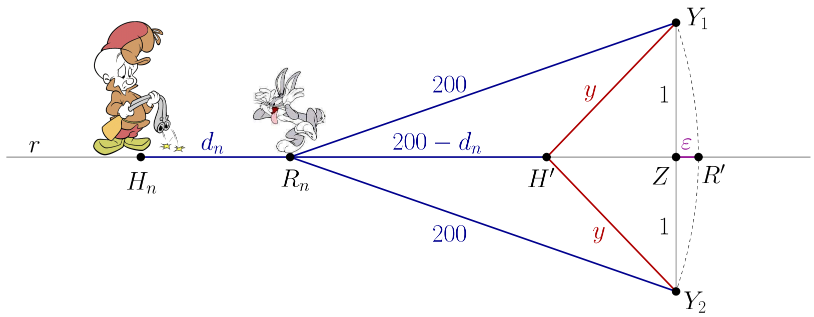
Answer: There is no such strategy for the hunter. The rabbit "wins".

Solution. If the answer were "yes", the hunter would have a strategy that would "work", no matter how the rabbit moved or where the radar pings R'_n appeared. We will show the opposite: with bad luck from the radar pings, there is no strategy for the hunter that guarantees that the distance stays below 100 in 10^9 rounds.

So, let d_n be the distance between the hunter and the rabbit after n rounds. Of course, if $d_n \geqslant 100$ for any $n < 10^9$, the rabbit has won — it just needs to move straight away from the hunter, and the distance will be kept at or above 100 thereon.

We will now show that, while $d_n < 100$, whatever given strategy the hunter follows, the rabbit has a way of increasing d_n^2 by at least $\frac{1}{2}$ every 200 rounds (as long as the radar pings are lucky enough for the rabbit). This way, d_n^2 will reach 10^4 in less than $2 \cdot 10^4 \cdot 200 = 4 \cdot 10^6 < 10^9$ rounds, and the rabbit wins.

Suppose the hunter is at H_n and the rabbit is at R_n. Suppose even that the rabbit *reveals* its position at this moment to the hunter (this allows us to ignore all information from previous radar pings). Let r be the line H_nR_n, and Y_1 and Y_2 be points which are 1 unit away from r and 200 units away from R_n, as in the figure below.



The rabbit's plan is simply to choose one of the points Y_1 or Y_2 and hop 200 rounds straight towards it. Since all hops stay within 1 distance unit from r, it is possible that all radar pings stay on r. In particular, in this case, the hunter has no way of knowing whether the rabbit chose Y_1 or Y_2.

Looking at such pings, what is the hunter going to do? If the hunter's strategy tells him to go 200 rounds straight to the right, he ends up at point H' in the figure. Note that the hunter does not have a better alternative! Indeed, after these 200 rounds he will always end up at a point to the left of H'. If his strategy took him to a point above r, he would end up even further from Y_2; and if his strategy took him below r, he would end up even further from Y_1. In other words, no matter what strategy the hunter follows, he can never be sure his distance to the rabbit will be less than $y \stackrel{\text{def}}{=} H'Y_1 = H'Y_2$ after these 200 rounds.

To estimate y^2, we take Z as the midpoint of segment Y_1Y_2, we take R' as a point 200 units to the right of R_n and we define $\varepsilon = ZR'$ (note that $H'R' = d_n$). Then

$$y^2 = 1 + (H'Z)^2 = 1 + (d_n - \varepsilon)^2$$

where

$$\varepsilon = 200 - R_n Z = 200 - \sqrt{200^2 - 1} = \frac{1}{200 + \sqrt{200^2 - 1}} > \frac{1}{400}.$$

In particular, $\varepsilon^2 + 1 = 400\varepsilon$, so

$$y^2 = d_n^2 - 2\varepsilon d_n + \varepsilon^2 + 1 = d_n^2 + \varepsilon(400 - 2d_n).$$

Since $\varepsilon > \frac{1}{400}$ and we assumed $d_n < 100$, this shows that $y^2 > d_n^2 + \frac{1}{2}$. So, as we claimed, with this list of radar pings, no matter what the hunter does, the rabbit might achieve $d_{n+200}^2 > d_n^2 + \frac{1}{2}$. The wabbit wins.

Comment 1. Many different versions of the solution above can be found by replacing 200 with some other number N for the number of hops the rabbit takes between reveals. If this is done, we have:

$$\varepsilon = N - \sqrt{N^2 - 1} > \frac{1}{N + \sqrt{N^2 - 1}} > \frac{1}{2N}$$

and

$$\varepsilon^2 + 1 = 2N\varepsilon,$$

so, as long as $N > d_n$, we would find

$$y^2 = d_n^2 + \varepsilon(2N - 2d_n) > d_n^2 + \frac{N - d_n}{N}.$$

For example, taking $N = 101$ is already enough—the squared distance increases by at least $\frac{1}{101}$ every 101 rounds, and $101^2 \cdot 10^4 = 1.0201 \cdot 10^8 < 10^9$ rounds are enough for the rabbit. If the statement is made sharper, some such versions might not work any longer.

Comment 2. The original statement asked whether the distance could be kept under 10^{10} in 10^{100} rounds.